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Analysis of Multiloop, Multirate Sampled-Data Systems

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Based upon new identities between z-transforms at a basis rate, z-transforms at faster rates, and modified z-transforms, the equivalence between the frequency decomposition method and the switch decomposition method is precisely presented so that results of one method are easily obtained from results of the other. Next, a method is developed for determining the closed loop transfer function of multiloop, multirate sampled-data systems with noninteger ratio sampling rates. Previously, this process involved solving a complex system of equations with rational polynomial coefficients. Herein, this is avoided by introducing a systematic decomposition of matrix operators which naturally arise from the switch decomposition method. The matrix operators are simplified by introducing the shifted transforms of signals sampled at one of the faster rates.

Introduction

THE use of digital computers in the guidance and control of missiles and space vehicles motivates the use of sampled data analysis methods. Such computers generally close several loops in the guidance and control tasks. Some of the guidance and control tasks can be performed more slowly than others while the over-all system meets all performance requirements. The use of multiple sampling rates can result in reduced computer capacity.

Previously developed methods of analysis of multiloop, multirate sampled-data systems are quite complex for systems which appear simple at first glance. Coffey and Williams[1] introduced a frequency-domain decomposition method for stability analysis of multiloop, multirate systems. In the general case, this frequency decomposition method numerically evaluated complex characteristic determinants for performing Nyquist stability analysis. Their analysis approach was a considerable improvement over Kranc's[2] switch decomposition method which was "state-of-the-art" up to that time. The switch decomposition method was based upon the frequency domain work of Sklansky[3] and was implemented by replacing samplers operating at various rates by an equivalent system of switches operating at a single rate. Jury[4] derived input-output relations for simple, open loop systems via both the frequency decomposition and switch decomposition methods and stated that the two forms are equivalent. The equivalence of the two forms could not be shown at that time due to a lack of invertible equivalence relations between the modified z-transform and the shifted argument, standard z-transforms. Boykin and Frazier[5] derived such equivalence relations and used these to develop a method of spectral factorization of the matrix operators which naturally arise in the switch decomposition method. In the case of sampling rates with integer ratios this spectral factorization enables the determination of closed loop transfer functions without ad hoc solutions of a system of equations with rational polynomial coefficients. The output was determined by inverting a spectrally factored matrix and was a one-step process.

This paper shows the equivalence of the frequency decomposition and switch decomposition methods through the equivalence relations and develops a method of determining closed loop transfer functions or characteristic equations of multiloop, multirate noninteger ratio sampled-data systems without evaluating

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a sequence of determinants or directly solving a system of equations with rational polynomial coefficients.

Notation, Z-Transform Identities and Equivalence Relations

When a system contains sampling elements which operate at various rates, confusion can arise unless the functional notation explicitly indicates the sampling period. Many authors (including Kranc[2] and Jury[4]) simply change the independent z-variable and retain the function name for distinctly different functions. With this in mind we submit the following notation.

The output of an impulse sampler operating at a sampling interval T_1 is denoted by $f^{*T_1}(t)$ if the input is $f(t)$. The Laplace transform of the output is $F^{*T_1}(s)$. The z-transform of the input $f(t)$ is

$$F^{T_1}(z_1) = Z_{T_1}[f(t)] = F^{*T_1}[T_1^{-1}\ln(z_1)]$$

where the complex logarithm of z_1 is the complex number sT_1 such that

$$e^{sT_1} = z_1$$

In terms of the z-transform variable the equation (A4) of Coffey and Williams is

$$A^T(z) = \frac{1}{n}\sum_{p=o}^{n-1} A^{T/n}(z_n W_n^{\pm p})$$

where $W_n^p = \exp(j2\pi p/n)$, $A^T(z)$ is the z-transform of a function $a(t)$ sampled at the slow rate of $1/T$ and $A^{T/n}(z_n W_n^p)$ is the shifted argument transform at the faster rate n/T.

The integer rate identity (IRI) and the rational rate identity (RRI) of Coffey and Williams can be expressed as

$$Z_{T/n}[g(t)h^{*T}(t)] = G^{T/n}(z_n)H^T(z) \quad \text{(IRI)} \tag{1}$$

$$Z_{T_1}[g(t)h^{*T_2}(t)] = \frac{1}{n_2}\sum_{m=o}^{n_2-1} G^{T_o}(z_o W_2^m)H^{T_2}(z_2 W_2^{mn_1}) \quad \text{(RRI)} \tag{2}$$

where $T_i = T/n_i$, $T_o = T/n_1 n_2$ and $z_o = e^{sT_o}$.

The equivalence relations, which are used in developments to follow, are the multiple rate identity (MRI)

$$A^T(z, k/n) = \frac{1}{n}\sum_{p=o}^{n-1} A^{T/n}(z_n W_n^p)\cdot(z_n^{-1}W_n^p)^{n-k} \qquad k = 0,1,\ldots,n-1 \tag{3}$$

and the inverse of the multiple rate identity (IMRI)

$$A^{T/n}(z_n W_n^k) = \sum_{p=o}^{n-1}(z_n W_n^k)^{p-n}A^T(z, p/n) \tag{4}$$

where $A^T(z, p/n)$ is the modified z-transform of sampling rate $1/T$. These equivalence relations were proven in another paper.[5]

Equivalence of Switch and Frequency Decomposition Methods

Equivalence is shown by: 1) establishing factors which occur in each of the switch and frequency decomposition methods and 2) showing that the possible combinations of factors for each method are equal.

To establish the factors, consider the simple example of Fig. 1a. The Laplace transform of the output can be expressed by

$$C^{*T}(s) = [G(s)R^{*T/n}(s)]^{*T} \tag{5}$$

Now apply the RRI, Eq. (2), to Eq. (5) and obtain the frequency decomposition expression

$$C^T(z) = \frac{1}{n}\sum_{m=o}^{n-1} G^{T/n}(z_n W_n^m)R^{T/n}(Z_n W_n^m) \tag{6}$$

To simplify the topological operations of the switch decomposition method and the resulting expressions we introduce[5] a vector of advances $\mathbf{E}^{n+}$ with components $E_i^{n+} = \exp[sT(i-1)/n]$, $i = 1,2,\ldots,n$, and a row vector of delays $\mathbf{E}^{n-}$ with components $E_i^{n-} = \exp[-sT(i-1)/n]$. Then, the equivalent switch decomposition system shown in Fig. 1b can be simply depicted by the flow graph of Fig. 1c. The switch decomposition representation of the output $C^T(z)$ for Fig. 1c is

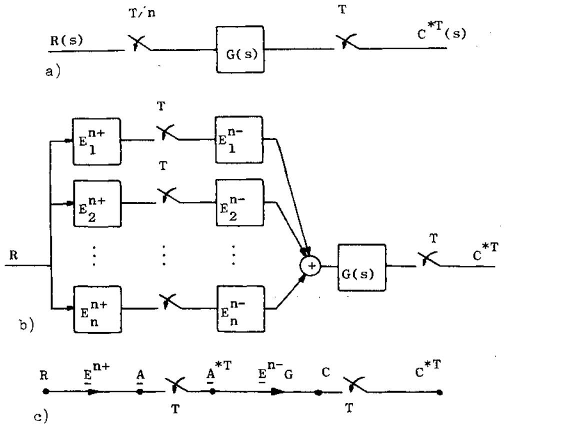

Fig. 1 Simple multirate example with integer ratio rates: a) block diagram; b) switch decomposition representation; c) matrix operator flow graph.

$$C^T(z) = (G\mathbf{E}^{n-})^T(\mathbf{E}^{n+}R)^T \tag{7}$$

The factors which appear in Eq. (7) are the two which naturally occur in transfer functions via the switch decomposition method. Operations with these factors can be simplified by introducing the vectors

$$\mathbf{e}_{kp} = \frac{1}{n_k^{1/2}}\begin{bmatrix}\lambda_{kp}^{o}\\ \lambda_{kp}^{1}\\ \vdots\\ \lambda_{kp}^{(n_k-1)}\end{bmatrix} \qquad \mathbf{v}_{kp} = \frac{1}{n_k^{1/2}}\begin{bmatrix}\lambda_{kp}^{o}\\ \lambda_{kp}^{-1}\\ \vdots\\ \lambda_{kp}^{-(n_k-1)}\end{bmatrix} \tag{8}$$

where $\lambda_{kp} = z^{1/n_k}W_k^p$, $p = 0,1,\ldots,n_k-1$. We now use the equivalence relations, Eqs. (3) and (4), to show that Eqs. (6) and (7) are equivalent. The use of Eq. (3) and the definitions of Eq. (8) enable us to write the factors of Eq. (7) as

$$(G\mathbf{E}^{n-})^T = \frac{1}{n^{1/2}}\sum_{p=o}^{n-1} G^{T/n}(z_n W_n^p)\mathbf{v}_{np}' \tag{9}$$

$$(\mathbf{E}^{n+}R)^T = \frac{1}{n^{1/2}}\sum_{p=o}^{n-1} R^{T/n}(z_n W_n^p)\mathbf{e}_{np} \tag{10}$$

so that the transfer function, Eq. (7), can be written

$$C^T(z) = \frac{1}{n}\sum_{m=o}^{n-1} G^{T/n}(z_n W_n^m)R^{T/n}(z_n W_n^m) \tag{11}$$

since $\mathbf{v}_{np}'\mathbf{e}_{nq} = \delta_{pq}$, the kronecker delta. Equation (11) is precisely the same as Eq. (6). In a similar way Eq. (7) can be derived from Eq. (6) via Eq. (4).

The previous, simple example contained one of the three types of matrix operator factors. The only other possible types of factors occur in the matrix operator representation of the switch decomposition formulation of the example depicted by Fig. 2a. By repetitive application of the RRI, Eq. (2), and use of the IRI, Eq. (1), we obtain the frequency decomposition representation

$$C^{T_3}(z_3) = \frac{R^T(z)}{n_1 n_2}\sum_{l=o}^{n_1-1}\sum_{m=o}^{n_2-1} G_1^{T_1}(z_1 W_1^{n_2 l+n_3 m}) \times G_2^{T_{12}}(z_{12}W_{12}^{n_2 l+n_3 m})G_3^{T_{23}}(z_{23}W_{23}^{n_3 m}) \tag{12}$$

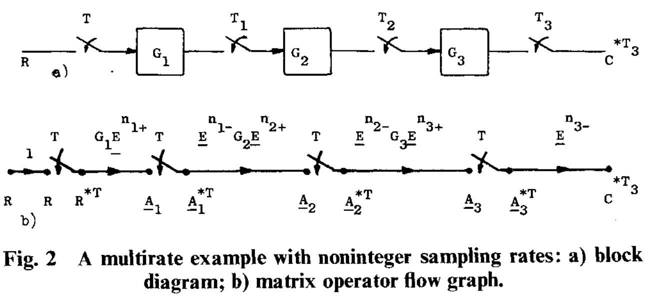

Fig. 2 A multirate example with noninteger sampling rates: a) block diagram; b) matrix operator flow graph.

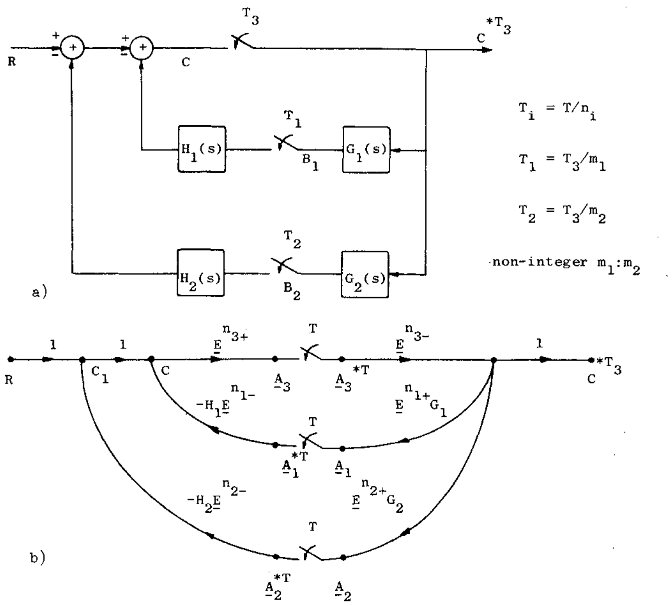

Fig. 3 A closed-loop example with noninteger ratio sampling rates: a) block diagram; b) matrix operator flow graph.

The matrix operator, switch decomposition representation obtained from Fig. 2(b) is

$$C^{T_3}(z_3) = R^T(z)\mathbf{E}^{n_3-}(\mathbf{E}^{n_3+}G_3\mathbf{E}^{n_2-})^T(\mathbf{E}^{n_2+}G_2\mathbf{E}^{n_1-})^T(\mathbf{E}^{n_1}G_1)^T \quad (13)$$

Again we use the equivalence relations, Eqs. (3) and (4), to show the equivalence of Eqs. (12) and (13). If we apply the definitions of Eq. (8) to Eq. (3), we discover that the rectangular matrix factors in Eq. (13) can be written as

$$(\mathbf{E}^{n_2+}G\mathbf{E}^{n_1-})^T = \frac{1}{(n_1n_2)^{1/2}} \sum_{p=o}^{n_1n_2-1} G^{T_{12}}(z_{12}W_{12}{}^p)\mathbf{e}_{2p}\mathbf{v}'_{1p} \quad (14)$$

where $T_{12} = T/n_1n_2$, $z_{12} = z^{1/n_1n_2}$ and $W_{12}{}^p = W_n{}^{np/n_1n_2}$. Since $\mathbf{E}^{n_3-} = n_3{}^{1/2}\mathbf{v}'_{30}$, the application of Eqs. (10) and (14) to Eq. (13) results in

$$C^{T_3}(z_3) = \frac{R^T(z)}{n_2n_1}\mathbf{v}'_{30} \sum_{p=o}^{n_2n_3-1} G_3{}^{T_{23}}(z_{23}W_{23}{}^p)\mathbf{e}_{3p}\mathbf{v}'_{2p} \times$$
$$\sum_{q=o}^{n_1n_2-1} G_2{}^{T_{12}}(z_{12}W_{12}{}^q)\mathbf{e}_{2q}\mathbf{v}'_{1q} \sum_{r=o}^{n_1-1} G_1{}^{T_1}(z_1W_1{}^r)\mathbf{e}_{1r} \quad (15)$$

The inner product $\mathbf{v}'_{30}\mathbf{e}_{3p} = 0$ except when it is unity for $p = 0, n_3, 2n_3, \ldots, (n_2-1)n_3$. Thus, Eq. (15) reduces to

$$C^{T_3}(z_3) = \frac{R^T(z)}{n_1n_2} \sum_{m=o}^{n_2-1} G_3{}^{T_{23}}(z_{23}W_{23}{}^{mn_3})\mathbf{v}'_{2,mn_3} \times$$
$$\sum_{q=o}^{n_1n_2-1} G_2{}^{T_{12}}(z_{12}W_{12}{}^q)\mathbf{e}_{2q}\mathbf{v}'_{1q} \sum_{r=o}^{n_1-1} G_1{}^{T_1}(z_1W_1{}^r)\mathbf{e}_{1r} \quad (16)$$

Since the inner product $\mathbf{v}'_{2,mn_3}\mathbf{e}_{2q} = 0$ except when it is unity for both $q = 0, 1, \ldots, n_1n_2-1$ and $q = n_3m+n_2n+kn_1n_2$ for integers n and k, we can write Eq. (16) equivalently as

$$C^{T_3}(z_3) = \frac{R^T(z)}{n_1n_2} \sum_{m=o}^{n_2-1} G_3{}^{T_{23}}(z_{23}W_{23}{}^{n_3m}) \sum_{n=o}^{n_1-1} G_2{}^{T_{12}} \times$$
$$(z_{12}W_{12}{}^{n_3m+n_2n+kn_1n_2})\mathbf{v}'_{1,n_3m+n_2n+kn_1n_2} \times$$
$$\sum_{r=o}^{n_1-1} G_1{}^{T_1}(z_1W_1{}^r)\mathbf{e}_{1r} \quad (17)$$

where k can be any integer. The integer k is of no consequence since $W_{ij}{}^{kn_in_j} = 1$. Finally, performing the inner products of $\mathbf{v}'_{1,n_3m+n_2n}$ with $\mathbf{e}_{1r}$, we reduce Eq. (17) to the frequency response representation, Eq. (12). These inner products have the values

$$\mathbf{v}'_{1,n_3m+n_2n}\mathbf{e}_{1r} = \begin{cases} 0 \text{ if } r-(n_3m+n_2n) \neq \text{int.} \times n_1 \\ 1 \text{ if } r-(n_3m+n_2n) = \text{int.} \times n_1 \end{cases}$$

and again we have used the fact that

$$W_1{}^{kn_1} = 1, \qquad k = \text{any integer}$$

In a similar way Eq. (13) can be derived from Eq. (12) via Eq. (4). The groups of factors which appear in the representations of the frequency and switch decomposition methods have been exhibited and shown to be equivalent.

Closed-Loop Response of Multirate Sampled Data Systems

The determination of the z-transformed, closed-loop response of a multirate sampled data system requires the solution of a set of simultaneous equations with rational polynomial coefficients. In this section we apply Eqs. (9, 10, and 14) to the factors which occur in the response relation via the vector operator method; and, by "spectrally factoring" the coefficient matrix, we obtain the closed-loop response.

The method is illustrated by considering a simple example with three samplers operating at different, noninteger ratio rates. Figure 3a depicts the example. It represents a practical, single axis, guidance and control problem. This example was used by Coffey and Williams.[1] Their frequency decomposition method resulted in a characteristic determinant with rational polynomial elements which was evaluated numerically to obtain a frequency response.

Figure 3b is the matrix operator flow graph representation of the system of Fig. 3a. By standard block diagram reduction methods and single rate operations we obtain

$$D\cdot(\mathbf{E}^{n_3+}C)^{*T} = (\mathbf{E}^{n_3+}R)^{*T} \quad (18)$$

where the matrix D is

$$D = I+(\mathbf{E}^{n_3+}H_1\mathbf{E}^{n_1-})^{*T}(\mathbf{E}^{n_1+}G_1\mathbf{E}^{n_3-})^{*T}+ (\mathbf{E}^{n_3+}H_2\mathbf{E}^{n_2-})^{*T}(\mathbf{E}^{n_2+}G_2\mathbf{E}^{n_3})^{*T}$$

and I is the $n_3 \times n_3$ identity matrix. If we pre-multiply both sides of Eq. (18) by the inverse of D, say D^{-1}, we obtain

$$(\mathbf{E}^{n_3+}C)^{*T} = D^{-1}(\mathbf{E}^{n_3+}R)^{*T} \quad (19)$$

We observe from Fig. 3b and Eq. (19) that

$$C^{*T_3} = \mathbf{E}^{n_3-}(\mathbf{E}^{n_3+}C)^{*T} = \mathbf{E}^{n_3-}D^{-1}(\mathbf{E}^{n_3+}R)^{*T} \quad (20)$$

Equation (20) represents the desired closed-loop response of the noninteger ratio, multirate, sampled data system.

The inverse of the matrix D is now determined in a systematic way. A typical term in the n_3-dimensional, D matrix can be expressed via Eq. (14) as

$$M_1 = (\mathbf{E}^{n_3+}H_1\mathbf{E}^{n_1-})^T(\mathbf{E}^{n_1+}G_1\mathbf{E}^{n_3-})^T = \frac{1}{n_1n_3} \sum_{p=o}^{n_1n_3-1} H_1{}^{T_{13}} \times$$
$$(z_{13}W_{13}{}^p)\mathbf{e}_{3p}\mathbf{v}'_{1p} \sum_{q=o}^{n_1n_3-1} G_1{}^{T_{13}}(z_{13}W_{13}{}^q)\mathbf{e}_{1q}\mathbf{v}'_{3q} \quad (21)$$

Many of the terms in Eq. (21) will be zero since the inner product $\mathbf{v}'_{1p}\mathbf{e}_{1q} = 0$ unless $q = p$ or p and q differ by an integer multiple of n_1. The reason that $\mathbf{v}'_{1p}\mathbf{e}_{1q} = 1$ when $p \neq q$ is the summation over p and q exceeds n_1 terms by an integer multiple, namely $\mathbf{n}_3$. Thus, if the integers l, m, and n have the ranges

$$l = 0, 1, \ldots, m_1-1$$
$$m = 0, 1, \ldots, n_3-1$$
$$n = 0, 1, \ldots, n_3-1$$

then p and q will have the correct ranges with the nonzero terms determined by

$$\mathbf{v}'_{1p}\mathbf{e}_{1q} = \delta_{p,l+mn_1}\cdot\delta_{q,l+nn_1} \quad (22)$$

where, for example,

$$\delta_{p,}\, l+mn_1 = \begin{cases} 1, & \text{if } p = l+mn_1 \\ 0, & \text{if } p \neq l+mn_1 \end{cases}$$

Substitution of Eq. (22) into Eq. (21) results in

$$M_1 = \frac{1}{n_1n_3} \sum_{l=o}^{n_1-1} \sum_{m=o}^{n_3-1} H_1{}^{T_{13}}(z_{13}W_{13}{}^{l+mn_1})\mathbf{e}_{3,l+mn_1} \times$$
$$\sum_{n=o}^{n_3-1} G_1{}^{T_{13}}(z_{13}W_{13}{}^{l+nn_1})\mathbf{v}'_{3,l+nn_1} \quad (23)$$

We observe that

$$\mathbf{e}_{3,l+mn_1} = \bar{W}_{31}(m)\mathbf{e}_{3l}$$
$$\mathbf{v}'_{3,l+nn_1} = \mathbf{v}'_{3l}\,\bar{W}_{31}{}^{-1}(n) \tag{24}$$

where

$$\bar{W}_{31}(k) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & W_3{}^{kn_1} & 0 & \cdots & 0 \\ 0 & 0 & W_3{}^{2kn_1} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & W_3{}^{(n_3-1)kn_1} \end{bmatrix}$$

With Eqs. (24) we can write Eq. (23) as

$$M_1 = \frac{1}{n_1 n_3}\sum_{l=o}^{n_1-1}\sum_{m=o}^{n_3-1} H_1{}^{T_{13}}(z_{13}W_{13}{}^{l+mn_1}) \times$$
$$\bar{W}_{31}(m)\mathbf{e}_{3l}\mathbf{v}'_{3l}\sum_{n=o}^{n_3-1} G_1{}^{T_{13}}(z_{13}W_{13}{}^{l+nn_1})\bar{W}_{31}{}^{-1}(n) \tag{25}$$

If in Eq. (25) we replace the index "1" with the index "2," we obtain the result for the other complex term in the D matrix of Eq. (20)—that is,

$$M_2 = (\mathbf{E}^{n_3+}H_2\mathbf{E}^{n_2-})^T(\mathbf{E}^{n_2+}G_2\mathbf{E}^{n_3-})^T =$$
$$\frac{1}{n_2 n_3}\sum_{l=o}^{n_2-1}\sum_{m=o}^{n_3-1} H_2{}^{T_{23}}(z_{23}W_{23}{}^{l+mn_2}) \times$$
$$\bar{W}_{32}(m)\mathbf{e}_{3l}\mathbf{v}'_{3l}\sum_{n=o}^{n_3-1} G_2{}^{T_{23}}(z_{23}W_{23}{}^{l+nn_2})\bar{W}_{32}{}^{-1}(n) \tag{26}$$

Now, for the closed-loop example with noninteger ratio sampling rates of $T_i = T/n_i$ with $n_1 = m_1 n_3$ and $n_2 = m_2 n_3$ for arbitrary integers m_1 and m_2, we can determine the response by carrying out the operations indicated in Eq. (10). We observe that

$$\bar{W}_{3i}(k) = I, \quad i = 1, 2, \quad k = \text{any integer}$$
$$\sum_{l=o}^{n_i-1} x_l = \sum_{q=o}^{n_3-1}\sum_{p=o}^{m_i-1} x_{q+pn_3}, \qquad i = 1,2$$
$$\mathbf{e}_{3,q+pn_3} = \mathbf{e}_{3q}$$
$$\mathbf{v}_{3,q+pn_3} = \mathbf{v}_{3q}$$

so that Eqs. (25) and (26) can be written

$$M_i = \frac{1}{n_i n_3}\sum_{q=o}^{n_3-1}\sum_{p=o}^{m_i-1}\sum_{m=o}^{n_3-1} H_i{}^{T_{i3}}(z_{i3}W_{i3}{}^{q+pn_3+mn_i}) \times$$
$$\mathbf{e}_{3q}\mathbf{v}'_{3q}\sum_{n=o}^{n_3-1} G_i{}^{T_{i3}}(z_{i3}W_{i3q+pn_3+mn_i}) \tag{27}$$

Also

$$I = \sum_{q=o}^{n_3-1}\mathbf{e}_{3q}\mathbf{v}'_{3q}$$

so that, using Eqs. (27), the matrix D becomes

$$D = \sum_{q=o}^{n_3-1}\lambda_q\mathbf{e}_{3q}\mathbf{v}'_{3q} \tag{28}$$

where

$$\lambda_q = 1 + \sum_{m=o}^{n_3-1}\sum_{n=o}^{n_3-1}\frac{1}{n_1 n_3}\sum_{p=o}^{m_1-1} H_1{}^{T_{13}}(z_{13}W_{13}{}^{q+pn_3+mn_1}) \times$$
$$G_1{}^{T_{13}}(z_{13}W_{13}{}^{q+pn_3+mn_1}) + \frac{1}{n_2 n_3}\sum_{r=o}^{m_2-1} H_2{}^{T_{23}} \times$$
$$(z_{23}W_{23}{}^{q+rn_3+mn_2})G_2{}^{T_{23}}(z_{23}W_{23}{}^{q+rn_3+nn_2})$$

This eigenvalue of the D matrix can be simplified by using the MRI, Eq. (3), with $n = k$ on the summations over m and n. The result is

$$\lambda_q = 1 + (1/m_1)\sum_{p=o}^{m_1-1} H_1{}^{T_1}(z_1W_1{}^{q+pn_3})G_1{}^{T_1}(z_1W_1{}^{q+pn_3}) +$$
$$(1/m_2)\sum_{r=o}^{m_2-1} H_2{}^{T_2}(z_2W_2{}^{q+rn_3})G_2{}^{T_2}(z_2W_2{}^{q+rn_3}) \tag{29}$$

The inverse of the D matrix is obtained from Eqs. (28) and (29) by substituting the reciprocal of λ_q from Eq. (29) for λ_q in Eq. (28)—that is,

$$D^{-1} = \sum_{q=o}^{n_3-1}\lambda_q{}^{-1}\mathbf{e}_{3q}\mathbf{v}'_{3q}$$

so that the response is determined from Eqs. (10, 20, 29, and 30) as

$$C^{T_3}(z_3) = \mathbf{E}^{n_3-}(z_3)D^{-1}(\mathbf{E}^{n_3+}R)^T = \mathbf{v}'_{3o}(z_3)\sum_{q=o}^{n_3-1}\lambda_q{}^{-1} \times$$
$$\mathbf{e}_{3q}\mathbf{v}'_{3q}\sum_{p=o}^{n_3-1} R^{T_3}(z_3W_3{}^p)\mathbf{e}_{3p} = R^{T_3}(z_3)/\lambda_o$$

Summary and Conclusions

A summary of the important multirate, sampled data identities were presented. Among these were our equivalence relations. We have shown that the terms which appear in z-transformed response functions by each of the switch decomposition and frequency decomposition methods are equivalent. The equivalence relations were used to show that terms obtained by one method are the same as the terms in the response function determined by the other method. Since the switch decomposition and frequency decomposition methods are equivalent, we could use either to determine a response function. However, the matrix operator flow graph with the multirate identity simplifies the structure and reduces the number of operations required.

A new method for determining the response function of a closed-loop, multirate, sampled-data system with noninteger ratio sampling rates was presented. This method was based upon a decomposition of our matrix operators. This leads to a well-structured transfer function matrix. In the noninteger ratio sampling rate example, this matrix was reduced to a spectrally factored form so that its inverse was obtained in one step. This offers a considerable savings in time and effort over the previously available methods which required special techniques for solving a system of simultaneous equations with rational polynomial coefficients.

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